Claim: Differential $\frac{dy}{dx} = \cos x + 3 \sin x$.

Proof:

1. **Claim:**
   \[ \frac{dy}{dx} = \cos x + 3 \sin x \]

2. **Proof:**
   
   Let $y(x) = \cos x + 3 \sin x$.

   Consider the function $y(x)$.

   - **Integral:**
     \[ \int \frac{dy}{dx} \, dx = 
       \begin{align*}
       & y(x) + C \\
       & = \cos x + 3 \sin x + C
       \end{align*} \]

   - **Differentiation:**
     \[ \frac{dy}{dx} = \cos x + 3 \sin x \]

   Therefore, \( \frac{dy}{dx} = \cos x + 3 \sin x \).
Positive homogeneity.

Definition. \( f \) is \( L \)-positive-homogeneous if \( \forall c \geq 0, \ f(c x) = c^L f(x) \).

Remarks. ReLU \( \sigma(x) = \max \{ 0, x \} \) is \( 1 \)-homogeneous
\[
\left( \sigma(c x) = \max \{ 0, c x \} = c \cdot \max \{ 0, x \} = c \sigma(x) \right).
\]

ReLU network is \( L \)-homogeneous in parameters:
\[
F(x; \mathbf{w}) = c^L \mathbf{W} \sigma(c W_1 \sigma(\ldots \sigma(c W_k x) \ldots))
\]
\[
= c^L W \sigma(W_1 \ldots W_k x \ldots) = c^L F(x; \mathbf{w}).
\]

Attention layer is not positive homogeneous.

Norns are \( 1 \)-homogeneous: \( \| c x \| = \| c \| \cdot \| x \| \).

Homogeneous polynomial: all terms have same degree (common degree is \( L \)).

Relationship to gradients.

Proposition. Suppose \( g \) is \( L \)-positive homogeneous & locally Lipschitz.
\( \forall w, \forall s \in \mathcal{R}(w) \) then \( \langle s, w \rangle = L \cdot g(w) \).

Examples.
\[
\langle \frac{2}{3} \| \mathbf{w} \|^2, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{w} \rangle = \| \mathbf{w} \|^2.
\]
\[
\text{sigmoid: } \sigma'(r) r = \begin{cases} r \leq 0 & \sigma(0) = 0 \Rightarrow \sigma(r) = 0 \\ r > 0 & \sigma(r) = 0 \end{cases}.
\]
ReLU: \( \sigma'(r) r = \begin{cases} r < 0 & 0 \cdot r = 0 = \sigma(r) \\ r > 0 & 0 \cdot r = 0 = \sigma(r) \\ r = 0 & \sigma(r) = r \end{cases} \).
ReLU network.